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# DEPTH OF INITIAL IDEALS OF NORMAL EDGE RINGS

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Abstract. Let $G$ be a finite graph on the vertex set $[d] = \{1, \ldots, d\}$ with the edges $e_1, \ldots, e_n$ and $K[\mathbf{t}] = K[t_1, \ldots, t_d]$ the polynomial ring in $d$ variables over a field $K$. The edge ring of $G$ is the semigroup ring $K[G]$ which is generated by those monomials $\mathbf{t}^e = t_i t_j$ such that $e = \{i, j\}$ is an edge of $G$. Let $K[\mathbf{x}] = K[x_1, \ldots, x_n]$ be the polynomial ring in $n$ variables over $K$ and define the surjective homomorphism $\pi : K[\mathbf{x}] \to K[G]$ by setting $\pi(x_i) = \mathbf{t}^{e_i}$ for $i = 1, \ldots, n$. The toric ideal $I_G$ of $G$ is the kernel of $\pi$. It will be proved that, given integers $f$ and $d$ with $6 \leq f \leq d$, there exist a finite connected nonbipartite graph $G$ on $[d]$ together with a reverse lexicographic order $<_{\text{rev}}$ on $K[\mathbf{x}]$ and a lexicographic order $<_{\text{lex}}$ on $K[\mathbf{x}]$ such that (i) $K[G]$ is normal, (ii) $\operatorname{depth} K[\mathbf{x}]/\operatorname{in}_{<_{\text{rev}}}(I_G) = f$ and (iii) $K[\mathbf{x}]/\operatorname{in}_{<_{\text{lex}}}(I_G)$ is Cohen–Macaulay, where $\operatorname{in}_{<_{\text{rev}}}(I_G)$ (resp. $\operatorname{in}_{<_{\text{lex}}}(I_G)$) is the initial ideal of $I_G$ with respect to $<_{\text{rev}}$ (resp. $<_{\text{lex}}$) and where $\operatorname{depth} K[\mathbf{x}]/\operatorname{in}_{<_{\text{rev}}}(I_G)$ is the depth of $K[\mathbf{x}]/\operatorname{in}_{<_{\text{rev}}}(I_G)$.

## Introduction

The study on edge rings [4] of finite graphs together with their toric ideals [5] have been achieved from viewpoints of both commutative algebra and combinatorics. Following the previous paper [3], which investigated a question about depth of edge rings, the topic of the present paper is depth of initial ideals of normal edge rings.

Let $G$ be a finite simple graph, i.e., a finite graph with no loop and no multiple edge, on the vertex set $[d] = \{1, \ldots, d\}$ and $E(G) = \{e_1, \ldots, e_n\}$ its edge set. Let $K[\mathbf{t}] = K[t_1, \ldots, t_d]$ be the polynomial ring in $d$ variables over a field $K$ and write $K[G]$ for the subring of $K[\mathbf{t}]$ generated by those monomials $\mathbf{t}^e = t_i t_j$ with $e = \{i, j\} \in E(G)$. The semigroup ring $K[G]$ is called the *edge ring* of $G$. Let $K[\mathbf{x}] = K[x_1, \ldots, x_n]$ be the polynomial ring in $n$ variables over $K$. The *toric ideal* of $G$ is the kernel $I_G$ of the surjective ring homomorphism $\pi : K[\mathbf{x}] \to K[G]$ defined by setting $\pi(x_i) = \mathbf{t}^{e_i}$ for $i = 1, \ldots, n$. Thus in particular one has $K[G] \cong K[\mathbf{x}]/I_G$. If $G$ is connected and nonbipartite (resp. connected and bipartite), then $\text{Krull-dim}\, K[G] = d$ (resp. $\text{Krull-dim}\, K[G] = d - 1$), where $\text{Krull-dim}\, K[G]$ stands for the Krull dimension of $K[G]$.

It follows from the criterion [4, Corollary 2.3] that the edge ring $K[G]$ of a connected graph $G$ is normal if and only if, for any two odd cycles $C_1$ and $C_2$ of $G$ having no common vertex, there exists an edge $\{v, w\}$ of $G$ such that $v$ is a vertex of $C_1$ and $w$ is a vertex of $C_2$.

We refer the reader to [2, Chapter 2] for fundamental materials on Gröbner bases. Let $<$ be a monomial order on $K[\mathbf{x}]$ and $\operatorname{in}_<(I_G)$ the initial ideal of $I_G$ with respect to $<$.

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The topic of this paper is $\operatorname{depth} K[\mathbf{x}]/\operatorname{in}_<(I_G)$, the depth of $K[\mathbf{x}]/\operatorname{in}_<(I_G)$, when $K[G]$ is normal. Computational experience yields the following

**Conjecture 0.1.** Let $G$ be a finite connected nonbipartite graph on $[d]$ with $d \geq 6$ and suppose that its edge ring $K[G]$ is normal. Then $\operatorname{depth} K[\mathbf{x}]/\operatorname{in}_<(I_G) \geq 6$ for any monomial order $<$ on $K[\mathbf{x}]$.

Now, even though Conjecture 0.1 is completely open, by taking Conjecture 0.1 into consideration, this paper will be devoted to proving the following

**Theorem 0.2.** *Given integers $f$ and $d$ with $6 \leq f \leq d$, there exists a finite connected nonbipartite graph $G$ on $[d]$ together with a reverse lexicographic order $<_{\text{rev}}$ on $K[\mathbf{x}]$ and a lexicographic order $<_{\text{lex}}$ on $K[\mathbf{x}]$ such that*

(i) $K[G]$ *is normal;*
(ii) $\operatorname{depth} K[\mathbf{x}]/\operatorname{in}_{<_{\text{rev}}}(I_G) = f$*;*
(iii) $K[\mathbf{x}]/\operatorname{in}_{<_{\text{lex}}}(I_G)$ *is Cohen–Macaulay.*

Let $k \geq 1$ be an arbitrary integer. We introduce the finite connected nonbipartite graph $G_{k+5}$ on $[k+5]$ which is drawn in Figure 0.1. Clearly, the edge ring $K[G_{k+5}]$ is normal. It will turn out that $G_{k+5}$ plays an important role in our proof of Theorem 0.2.

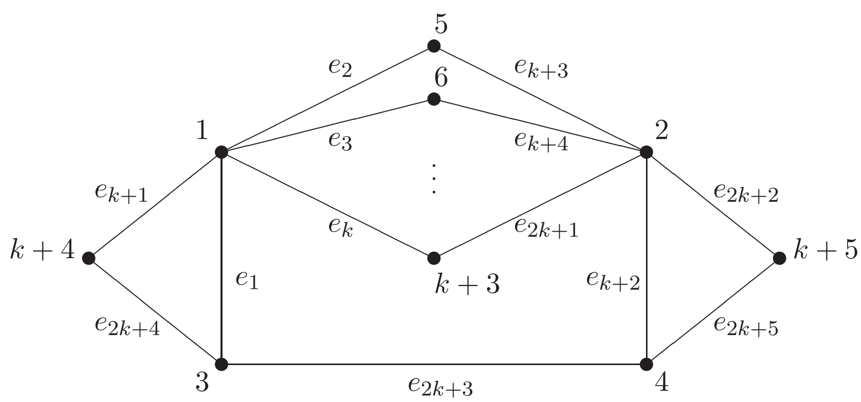


**Figure 0.1.** (finite graph $G_{k+5}$)

The essential step in order to prove Theorem 0.2 is to show the following

**Lemma 0.3.** *Let $<_{\text{rev}}$ (resp. $<_{\text{lex}}$) denote the reverse lexicographic order (resp. the lexicographic order) on $K[\mathbf{x}] = K[x_1, \ldots, x_{2k+5}]$ induced by the ordering $x_1 > \cdots > x_{2k+5}$ of the variables. Then*

(i) $\operatorname{depth} K[\mathbf{x}]/\operatorname{in}_{<_{\text{rev}}}(I_{G_{k+5}}) = 6$*;*
(ii) $K[\mathbf{x}]/\operatorname{in}_{<_{\text{lex}}}(I_{G_{k+5}})$ *is Cohen–Macaulay.*

Once we establish Lemma 0.3, to prove Theorem 0.2 is straightforward. In fact, given integers $f$ and $d$ with $6 \leq f \leq d$, we define the finite graph $\Gamma$ on $[d-f+6]$ to be $G_{d-f+6}$ with the edges $e_1, e_2, \ldots, e_{2(d-f)+7}$ and then introduce the finite connected nonbipartite graph $G$ on $[d]$ which is obtained from $\Gamma$ by adding $f-6$ edges

$$e_{2(d-f)+7+i} = \{1, d-f+6+i\}, \quad i = 1, \ldots, f-6$$

to $\Gamma$. Clearly, both edge rings $K[\Gamma]$ and $K[G]$ are normal, and

$$I_G = I_\Gamma(K[x_1, \dots, x_{2d-f+1}]).$$

Thus in particular

$$\mathrm{in}_<(I_G) = \mathrm{in}_<(I_\Gamma)(K[x_1, \dots, x_{2d-f+1}]),$$

where $<$ is any monomial order on $K[x_1, \dots, x_{2d-f+1}]$. Thus Lemma 0.3 guarantees that

$$\operatorname{depth} K[x_1, \dots, x_{2d-f+1}]/\,\mathrm{in}_{<_{\mathrm{rev}}}(I_G) = f$$

and $K[x_1, \dots, x_{2d-f+1}]/\,\mathrm{in}_{<_{\mathrm{lex}}}(I_G)$ is Cohen–Macaulay, as desired.

## 1. Preliminaries

Let $G = G_{k+5}$. In this section, we will find a Gröbner basis of $I_G$ and a set of generators of the initial ideal of $I_G$ with respect to the reverse lexicographic order.

Let $K[\mathbf{x}] = K[x_1, \dots, x_{2k+5}]$ be the polynomial ring in $2k+5$ variables over a field $K$. There are 4 kinds of primitive even closed walks of $G$:

(1) a 4-cycle: $(e_i, e_{k+1+i}, e_{k+1+j}, e_j)$, where $2 \le i < j \le k$;
(2) a walk on two 3-cycles and the same edge $e_{2k+3}$ combining two cycles: $(e_1, e_{k+1}, e_{2k+4}, e_{2k+3}, e_{k+2}, e_{2k+2}, e_{2k+5}, e_{2k+3})$;
(3) a 6-cycle: $(e_i, e_{k+1+i}, e_{k+2}, e_{2k+3}, e_{2k+4}, e_{k+1})$ or $(e_i, e_{k+1+i}, e_{2k+2}, e_{2k+5}, e_{2k+3}, e_1)$, where $2 \le i \le k$;
(4) a walk on two 3-cycles and the length 2 paths combining two cycles: $(e_{k+2}, e_{2k+5}, e_{2k+2}, e_{k+1+i}, e_i, e_1, e_{2k+4}, e_{k+1}, e_j, e_{k+1+j})$, where $2 \le i \le j \le k$.

It was proved in [5, Lemma 3.1] that the binomials corresponding to these primitive even closed walks generate the toric ideal $I_G$. Let $<_{\mathrm{rev}}$ be the reverse lexicographic order with $x_1 > x_2 > \cdots > x_{2k+5}$.

**Lemma 1.1.** *The set of binomials corresponding to primitive even closed walks (1), (2), (3) and (4) is a Gröbner basis of $I_G$ with respect to $<_{\mathrm{rev}}$.*

*Proof.* The result follows from a direct application of Buchberger's criterion to the set of generators of $I_G$ corresponding to the primitive even closed walks listed above. Let $f$ and $g$ be two such generators. We will prove that the $S$-polynomial $S(f,g)$ will reduce to 0 by generators of type (1), (2), (3) and (4). Let $i, j, p, q$ be integers with $2 \le i, j, p, q \le k$. On the following proof, we will underline the leading monomial of a binomial with respect to $<_{\mathrm{rev}}$.

**Case 1:** Let $f = x_i x_{k+1+j} - \underline{x_j x_{k+1+i}}$ and $g = x_p x_{k+1+q} - \underline{x_q x_{k+1+p}}$ be generators of type (1), where $i < j$ and $p < q$. If $i \neq p$ and $j \neq q$, then the leading monomials of $f$ and $g$ are coprime. Thus $S(f,g)$ will reduce to 0. We assume that $i = p$. Then

$$\begin{aligned} S(f,g) &= \frac{\mathrm{lcm}(LM_{<_{\mathrm{rev}}}(f), LM_{<_{\mathrm{rev}}}(g))}{LT_{<_{\mathrm{rev}}}(f)} f - \frac{\mathrm{lcm}(LM_{<_{\mathrm{rev}}}(f), LM_{<_{\mathrm{rev}}}(g))}{LT_{<_{\mathrm{rev}}}(g)} g \\ &= -x_q(x_i x_{k+1+j} - x_j x_{k+1+i}) - (-x_j)(x_i x_{k+1+q} - x_q x_{k+1+i}) \\ &= -x_i(x_q x_{k+1+j} - x_j x_{k+1+q}). \end{aligned}$$

Note that, up to sign, $x_q x_{k+1+j} - x_j x_{k+1+q}$ is a generator of $I_G$ of type (1). Therefore $S(f,g)$ will reduce to 0. The case of $j = q$ is similar.

**Case 2:** Let $f$ be the same as above and $g = x_1x_{k+2}x_{2k+4}x_{2k+5} - \underline{x_{k+1}x_{2k+2}x_{2k+3}^2}$ a generator of type (2). Since $2 \leq i < j \leq k$, the leading monomials of $f$ and $g$ are always coprime.
**Case 3:** Again, we set that $f$ is the same as above. Let $g$ be of type (3). First, let $g = x_px_{k+2}x_{2k+4} - \underline{x_{k+1}x_{k+1+p}x_{2k+3}}$. If $i \neq p$, then the leading monomials of $f$ and $g$ are coprime. We assume that $i = p$. Then

$$\begin{aligned} S(f,g) &= -x_{k+1}x_{2k+3}f - (-x_j)g \\ &= -x_ix_{k+1}x_{k+1+j}x_{2k+3} + x_ix_jx_{k+2}x_{2k+4} \\ &= x_i(x_jx_{k+2}x_{2k+4} - x_{k+1}x_{k+1+j}x_{2k+3}), \end{aligned}$$

where $x_jx_{k+2}x_{2k+4} - x_{k+1}x_{k+1+j}x_{2k+3}$ is of type (3). Next, let $g = \underline{x_px_{2k+2}x_{2k+3}} - x_1x_{k+1+p}x_{2k+5}$. If $j \neq p$, then the leading monomials of $f$ and $g$ are coprime. We assume that $j = p$. Then

$$\begin{aligned} S(f,g) &= -x_{2k+2}x_{2k+3}f - x_{k+1+i}g \\ &= -x_{k+1+j}(x_ix_{2k+2}x_{2k+3} - x_1x_{k+1+i}x_{2k+5}) \end{aligned}$$

and again we have that $x_ix_{2k+2}x_{2k+3} - x_1x_{k+1+i}x_{2k+5}$ is of type (3).
**Case 4:** Again, we assume that $f$ is the same as above. Let $g = \underline{x_px_qx_{k+2}x_{2k+2}x_{2k+4}} - x_1x_{k+1}x_{k+1+p}x_{k+1+q}x_{2k+5}$ be of type (4), where $p \leq q$. If $j \neq p$ and $j \neq q$, then the leading monomials of $f$ and $g$ are coprime. If $j = p$, then

$$\begin{aligned} S(f,g) &= -x_qx_{k+2}x_{2k+2}x_{2k+4}f - x_{k+1+i}g \\ &= -x_{k+1+j}(x_ix_qx_{k+2}x_{2k+2}x_{2k+4} - x_1x_{k+1}x_{k+1+i}x_{k+1+q}x_{2k+5}), \end{aligned}$$

which is a multiple of type (4) generator. The case of $j = q$ is similar.
**Case 5:** Let $f = x_1x_{k+2}x_{2k+4}x_{2k+5} - \underline{x_{k+1}x_{2k+2}x_{2k+3}^2}$ be a generator of type (2), and $g$ a generator of type (3). First we consider the case where $g = x_px_{k+2}x_{2k+4} - \underline{x_{k+1}x_{k+1+p}x_{2k+3}}$. Then

$$\begin{aligned} S(f,g) &= -x_{k+1+p}f - (-x_{2k+2}x_{2k+3})g \\ &= x_{k+2}x_{2k+4}(x_px_{2k+2}x_{2k+3} - x_1x_{k+1+p}x_{2k+5}), \end{aligned}$$

where $x_px_{2k+2}x_{2k+3} - x_1x_{k+1+p}x_{2k+5}$ is of type (3). Next, let $g = \underline{x_px_{2k+2}x_{2k+3}} - x_1x_{k+1+p}x_{2k+5}$. Then

$$\begin{aligned} S(f,g) &= -x_pf - x_{k+1}x_{2k+3}g \\ &= -x_1x_{2k+5}(x_px_{k+2}x_{2k+4} - x_{k+1}x_{k+1+p}x_{2k+3}) \end{aligned}$$

and we have that $x_px_{k+2}x_{2k+4} - x_{k+1}x_{k+1+p}x_{2k+3}$ is of type (3).
**Case 6:** Let $f$ be the same as in Case 5 and $g = \underline{x_px_qx_{k+2}x_{2k+2}x_{2k+4}} - x_1x_{k+1}x_{k+1+p}x_{k+1+q}x_{2k+5}$ be of type (4) generator, where $p \leq q$. Then

$$\begin{aligned} S(f,g) &= -x_px_qx_{k+2}x_{2k+4}f - x_{k+1}x_{2k+3}^2g \\ &= -x_1x_{2k+5}(x_px_qx_{k+2}^2x_{2k+4}^2 - \underline{x_{k+1}^2x_{k+1+p}x_{k+1+q}x_{2k+3}^2}) \\ &= -x_1x_{2k+5}\{x_{k+1}x_{k+1+q}x_{2k+3}(x_px_{k+2}x_{2k+4} - x_{k+1}x_{k+1+p}x_{2k+3}) \\ &\qquad +x_px_{k+2}x_{2k+4}(x_qx_{k+2}x_{2k+4} - x_{k+1}x_{k+1+q}x_{2k+3})\}. \end{aligned}$$

Thus $S(f,g)$ reduce to 0 by generators of type (3).
**Case 7:** We assume that both $f$ and $g$ are of type (3). First, we consider the case where $f = x_i x_{k+2} x_{2k+4} - \underline{x_{k+1} x_{k+1+i} x_{2k+3}}$ and $g = x_p x_{k+2} x_{2k+4} - \underline{x_{k+1} x_{k+1+p} x_{2k+3}}$, where $i \neq p$. Then

$$\begin{aligned} S(f,g) &= -x_{k+1+p} f - (-x_{k+1+i}) g \\ &= -x_{k+2} x_{2k+4} (x_i x_{k+1+p} - x_p x_{k+1+i}), \end{aligned}$$

which is a multiple of type (1) generator. Next, let $f$ be the same one and $g = \underline{x_p x_{2k+2} x_{2k+3}} - x_1 x_{k+1+p} x_{2k+5}$. Then

$$\begin{aligned} S(f,g) &= -x_p x_{2k+2} f - x_{k+1} x_{k+1+i} g \\ &= -(x_i x_p x_{k+2} x_{2k+2} x_{2k+4} - x_1 x_{k+1} x_{k+1+i} x_{k+1+p} x_{2k+5}), \end{aligned}$$

which is a generator of type (4) up to sign. Finally, let $f = \underline{x_i x_{2k+2} x_{2k+3}} - x_1 x_{k+1+i} x_{2k+5}$ and $g = \underline{x_p x_{2k+2} x_{2k+3}} - x_1 x_{k+1+p} x_{2k+5}$, where $i \neq p$. Then

$$\begin{aligned} S(f,g) &= x_p f - x_i g \\ &= x_1 x_{2k+5} (x_i x_{k+1+p} - x_p x_{k+1+i}). \end{aligned}$$

**Case 8:** Let $f$ be of type (3) and $g = \underline{x_p x_q x_{k+2} x_{2k+2} x_{2k+4}} - x_1 x_{k+1} x_{k+1+p} x_{k+1+q} x_{2k+5}$ be of type (4) with $p \leq q$. First, we set that $f = x_i x_{k+2} x_{2k+4} - \underline{x_{k+1} x_{k+1+i} x_{2k+3}}$. Then the leading monomials of $f$ and $g$ are coprime. Next, we set that $f = \underline{x_i x_{2k+2} x_{2k+3}} - x_1 x_{k+1+i} x_{2k+5}$. If $i \neq p$ and $i \neq q$, then

$$\begin{aligned} S(f,g) &= x_p x_q x_{k+2} x_{2k+4} f - x_i x_{2k+3} g \\ &= x_1 x_{2k+5} (\underline{x_i x_{k+1} x_{k+1+p} x_{k+1+q} x_{2k+3}} - x_p x_q x_{k+2} x_{k+1+i} x_{2k+4}) \\ &= x_1 x_{2k+5} \{ -x_i x_{k+1+q} (x_p x_{k+2} x_{2k+4} - x_{k+1} x_{k+1+p} x_{2k+3}) \\ &\qquad + x_p x_{k+2} x_{2k+4} (x_i x_{k+1+q} - x_q x_{k+1+i}) \}. \end{aligned}$$

Thus $S(f,g)$ reduce to 0 by generators of type (1) and (3). If $i = p$, then

$$\begin{aligned} S(f,g) &= x_q x_{k+2} x_{2k+4} f - x_{2k+3} g \\ &= -x_1 x_{k+1+i} x_{2k+5} (x_q x_{k+2} x_{2k+4} - x_{k+1} x_{k+1+q} x_{2k+3}). \end{aligned}$$

The case of $i = q$ is similar.
**Case 9:** Finally, we consider the case that both $f$ and $g$ are of type (4). Let $f = \underline{x_i x_j x_{k+2} x_{2k+2} x_{2k+4}} - x_1 x_{k+1} x_{k+1+i} x_{k+1+j} x_{2k+5}$ and
$g = \underline{x_p x_q x_{k+2} x_{2k+2} x_{2k+4}} - x_1 x_{k+1} x_{k+1+p} x_{k+1+q} x_{2k+5}$, where $i \leq j$ and $p \leq q$. Without loss of generality, we may assume that $j \geq q$. First, we assume that $j > q (\geq p)$. If $i \neq p$ and $i \neq q$, then

$$\begin{aligned} S(f,g) &= x_p x_q f - x_i x_j g \\ &= x_1 x_{k+1} x_{2k+5} (\underline{x_i x_j x_{k+1+p} x_{k+1+q}} - x_p x_q x_{k+1+i} x_{k+1+j}) \\ &= x_1 x_{k+1} x_{2k+5} \{ -x_i x_{k+1+q} (x_p x_{k+1+j} - x_j x_{k+1+p}) + x_p x_{k+1+j} (x_i x_{k+1+q} - x_q x_{k+1+i}) \}. \end{aligned}$$

Thus we have that $S(f,g)$ reduce to 0 by generators of type (1). If $i = p$, then

$$\begin{aligned} S(f,g) &= x_q f - x_j g \\ &= x_1 x_{k+1} x_{k+1+i} x_{2k+5} (x_j x_{k+1+q} - x_q x_{k+1+j}). \end{aligned}$$

The case of $i = q$ is similar. Next, we consider the case where $j = q$. Then $i \neq p$ and

$$\begin{aligned} S(f,g) &= x_p f - x_i g \\ &= x_1 x_{k+1} x_{k+1+j} x_{2k+5} (x_i x_{k+1+p} - x_p x_{k+1+i}), \end{aligned}$$

which is a multiple of type (1) generator. $\square$

**Corollary 1.2.** *The initial ideal of $I_G$ with respect to $<_{\text{rev}}$ is generated by the following monomials:*

$$\begin{aligned} & x_j x_{k+1+i}, \quad 2 \leq i < j \leq k, \\ & x_{k+1} x_{2k+2} x_{2k+3}^2, \\ & x_{k+1} x_{k+1+r} x_{2k+3}, \ x_r x_{2k+2} x_{2k+3}, \quad 2 \leq r \leq k, \\ & x_p x_q x_{k+2} x_{2k+2} x_{2k+4}, \quad 2 \leq p \leq q \leq k. \end{aligned}$$

*Proof.* By Lemma 1.1, the set of the binomials corresponding to primitive even closed walks (1), (2), (3) and (4) is a Gröbner basis of $I_G$ with respect to $<_{\text{rev}}$. Thus the leading terms with respect to $<_{\text{rev}}$ generate the initial ideal of $I_G$. $\square$

For the rest part of this paper, we will denote by $I$, the initial ideal of $I_G$ with respect to $<_{\text{rev}}$.

## 2. Proof of $\operatorname{depth} K[\mathbf{x}]/I \leq 6$

In this section, we will prove that $\operatorname{depth} K[\mathbf{x}]/I \leq 6$. Since the number of edges of $G$, which coincides with $2k+5$, is equal to the number of variables of $K[\mathbf{x}]$, Auslander–Buchsbaum formula implies that we may prove that $\operatorname{pd} K[\mathbf{x}]/I \geq 2k-1$, where $\operatorname{pd} K[\mathbf{x}]/I$ stands for the projective dimension of $K[\mathbf{x}]/I$.

First, we recall from [1] the fundamental technique to compute the Betti numbers of (non-squarefree) monomial ideals.

For a multi degree $\mathbf{a} \in \mathbb{Z}^n_{\geq 0}$, define

$$\mathbf{K}^{\mathbf{a}}(J) = \{\text{squarefree vectors } \alpha : \mathbf{x}^{\mathbf{a}-\alpha} \in J\}$$

to be the *Koszul simplicial complex* of $J$ in degree $\mathbf{a}$, where a squarefree vector $\alpha$ means that each entry of $\alpha$ is 0 or 1.

**Lemma 2.1.** ([1, Theorem 1.34]) *Let $S$ be a polynomial ring, $J$ a monomial ideal of $S$ and $\mathbf{a} \in \mathbb{Z}^n_{\geq 0}$ a vector. Then the Betti numbers of $J$ and $S/J$ in degree $\mathbf{a}$ can be expressed as*

$$\beta_{i,\mathbf{a}}(J) = \beta_{i+1,\mathbf{a}}(S/J) = \dim_K \tilde{H}_{i-1}(\mathbf{K}^{\mathbf{a}}(J); K).$$

By virtue of Lemma 2.1, in order to prove that $\operatorname{pd} K[\mathbf{x}]/I \geq 2k-1$, we may show the following

**Lemma 2.2.** *Let* $\mathbf{a} = \sum_{j=2}^{k}(\mathbf{e}_j + \mathbf{e}_{k+1+j}) + \mathbf{e}_{k+1} + \mathbf{e}_{2k+2} + 2\mathbf{e}_{2k+3} \in \mathbb{Z}_{\geq 0}^{2k+5}$*, where* $\mathbf{e}_i \in \mathbb{R}^{2k+5}$ *is the* $i$*-th unit vector of* $\mathbb{R}^{2k+5}$*. Then*

$$\dim_K \tilde{H}_{2k-3}(\mathbf{K}^{\mathbf{a}}(I); K) \neq 0.$$

*Proof.* Let $\Delta$ be the simplicial complex on the vertex set $[2k+5]$ which is obtained by identifying a squarefree vector $\alpha \in \mathbf{K}^{\mathbf{a}}(I)$ with the set of coordinates where the entries of $\alpha$ are 1. To prove the assertion, we may show that $\dim_K \tilde{H}_{2k-3}(\Delta; K) \neq 0$. Let $I_1$ (resp. $I_2$) be the monomial ideal generated by the monomials

$$\begin{aligned} &x_j x_{k+1+i}, \quad 2 \leq i < j \leq k, \\ &x_{k+1}x_{k+1+r}x_{2k+3}, \ x_r x_{2k+2} x_{2k+3}, \quad 2 \leq r \leq k \end{aligned}$$

(resp. by the monomial $x_{k+1}x_{2k+2}x_{2k+3}^2$). We denote by $\Delta_1, \Delta_2$, the subcomplexes of $\Delta$ corresponding to $\mathbf{K}^{\mathbf{a}}(I_1), \mathbf{K}^{\mathbf{a}}(I_2)$, respectively. Since the $(k+2)$-th entry of $\mathbf{a}$ is equal to 0, one has $\Delta = \Delta_1 \cup \Delta_2$. Moreover, one can verify that all the facets of $\Delta_1$ contain a common vertex $2k+3$. In other words, $\Delta_1$ is a cone over some simplicial complex. In addition, $\Delta_2$ has only one facet

$$\{2, 3, \ldots, k, k+3, k+4, \ldots, 2k+1\},$$

which is a $(2k-3)$-dimensional simplex. Thus the reduced homologies of both of $\Delta_1$ and $\Delta_2$ all vanish. Hence the Mayer–Vietoris sequence

$$\begin{aligned} \cdots \longrightarrow \tilde{H}_i(\Delta_1 \cap \Delta_2; K) \longrightarrow \tilde{H}_i(\Delta_1; K) \oplus \tilde{H}_i(\Delta_2; K) \longrightarrow \tilde{H}_i(\Delta; K) \\ \longrightarrow \tilde{H}_{i-1}(\Delta_1 \cap \Delta_2; K) \longrightarrow \tilde{H}_{i-1}(\Delta_1; K) \oplus \tilde{H}_{i-1}(\Delta_2; K) \longrightarrow \cdots \end{aligned}$$

yields

$$\tilde{H}_i(\Delta; K) \cong \tilde{H}_{i-1}(\Delta_1 \cap \Delta_2; K) \qquad \text{for all } i.$$

Now we note that subsets

$$\begin{aligned} &\{2, 3, \ldots, k, k+3, k+4, \ldots, 2k+1\} \setminus \{i\}, \qquad i = 2, \ldots, k, \\ &\{2, 3, \ldots, k, k+3, k+4, \ldots, 2k+1\} \setminus \{k+1+j\}, \qquad j = 2, \ldots, k \end{aligned}$$

are faces of $\Delta_1$ and $\{2, 3, \ldots, k, k+3, k+4, \ldots, 2k+1\}$ is not a face of $\Delta_1$. Thus the above subsets are the facets of $\Delta_1 \cap \Delta_2$. In particular, one has $\dim(\Delta_1 \cap \Delta_2) = 2k-4$. Since $\Delta_1 \cap \Delta_2$ contains all facets of the $(2k-3)$-dimensional simplex $\Delta_2$, the geometric realization of $\Delta_1 \cap \Delta_2$ is homeomorphic to the boundary complex of the simplex $\Delta_2$, i.e., $\Delta_1 \cap \Delta_2$ is a simplicial $(2k-4)$-sphere.

Therefore, one has $\dim_K \tilde{H}_{2k-3}(\Delta; K) = \dim_K \tilde{H}_{2k-4}(\Delta_1 \cap \Delta_2; K) \neq 0$. □

## 3. Proof of $\operatorname{depth} K[\mathbf{x}]/I \geq 6$

In this section, we will prove the following

**Lemma 3.1.**

$$\operatorname{depth} K[\mathbf{x}]/I \geq 6.$$

Before proving Lemma 3.1, we prepare the following two lemmas.

Let $J \subset S = K[x_1, \ldots, x_n]$ be a monomial ideal. We denote by $G(J)$, the minimal set of monomial generators of $J$.

**Lemma 3.2.** *Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring in $n$ variables and $J \subset S$ a monomial ideal of $S$.*

(a) *If only $m(\leq n)$ variables appear in the elements of $G(J)$, then* $\operatorname{depth} S/J \geq n - m$.

(b) *If only $m$ variables appear in the elements of $G(J)$ and the variables $x_{i_1}, \ldots, x_{i_r}$ do not appear in there, then* $\operatorname{depth} S/J' \geq n - m$*, where $J' = x_{i_1} \cdots x_{i_r} J$.*

*Proof.* Without loss of generality, we may assume that only the variables $x_1, \ldots, x_m$ appear in the elements of $G(J)$.
(a) Since the variables $x_{m+1}, \ldots, x_n$ do not appear in the elements of $G(J)$, the sequence $x_{m+1}, \ldots, x_n$ is an $S/J$-regular sequence. Thus one has $\operatorname{depth} S/J \geq n - m$.
(b) Set $x_{i_\ell} = x_{m+\ell}$ for $\ell = 1, \ldots, r$ and $J'' = (x_{m+1} \cdots x_{m+r}) \subset S$. Then, by the short exact sequence $0 \to S/J \cap J'' \to S/J \oplus S/J'' \to S/(J + J'') \to 0$, we have

$$\operatorname{depth} S/J' = \operatorname{depth} S/J \cap J'' \geq \min\{\operatorname{depth} S/J, \operatorname{depth} S/J'', \operatorname{depth} S/(J + J'') + 1\}.$$

Now we have $\operatorname{depth} S/J \geq n - m$ by (a) and $\operatorname{depth} S/J'' = n - 1$. In addition, since $x_{m+1}, \ldots, x_{m+r}$ do not appear in the elements of $G(J)$, the monomial $x_{m+1} \cdots x_{m+r}$ is an $S/J$-regular element. Hence one has $\operatorname{depth} S/(J + J'') = \operatorname{depth} S/J - 1 \geq n - m - 1$. $\square$

Let

$$\begin{aligned}
I_1 &= (x_j x_{k+1+i} : 2 \leq i < j \leq k), \\
I_2 &= (x_{k+1} x_{2k+2} x_{2k+3}^2), \\
I_3 &= x_{2k+2} x_{2k+3} (x_2, x_3, \ldots, x_k), \\
I_4 &= x_{k+1} x_{2k+3} (x_{k+3}, x_{k+4}, \ldots, x_{2k+1}), \\
I_5 &= x_{k+2} x_{2k+2} x_{2k+4} (x_2, x_3, \ldots, x_k)^2.
\end{aligned}$$

Then $I = I_1 + I_2 + \cdots + I_5$.

The following lemma can be obtained by elementary computations.

**Lemma 3.3.** *Let $J_1 = I_3 + I_4$, $J_2 = J_1 + I_1$ and $J_3 = J_2 + I_5$. Then*
(a) $I_3 \cap I_4 = x_{k+1} x_{2k+2} x_{2k+3} (x_2, \ldots, x_k)(x_{k+3}, \ldots, x_{2k+1})$.
(b) $J_1 \cap I_1 = x_{2k+3} (x_{k+1}, x_{2k+2}) I_1$.
(c) $J_2 \cap I_5 = x_{k+2} x_{2k+2} x_{2k+4} (x_2, \ldots, x_k)(x_{2k+3}(x_2, \ldots, x_k) + I_1)$.
(d) $J_3 \cap I_2 = x_{k+1} x_{2k+2} x_{2k+3}^2 (x_2, \ldots, x_k, x_{k+3}, \ldots, x_{2k+1})$.

Now we will prove Lemma 3.1.

*Proof of Lemma 3.1.* Work with the same notations as in Lemma 3.3. By the short exact sequence

$$0 \to K[\mathbf{x}]/J_3 \cap I_2 \to K[\mathbf{x}]/J_3 \oplus K[\mathbf{x}]/I_2 \to K[\mathbf{x}]/(J_3 + I_2) \to 0,$$

one has

$$\begin{aligned}
\operatorname{depth} K[\mathbf{x}]/I &= \operatorname{depth} K[\mathbf{x}]/(J_3 + I_2) \\
&\geq \min\{\operatorname{depth} K[\mathbf{x}]/J_3, \operatorname{depth} K[\mathbf{x}]/I_2, \operatorname{depth} K[\mathbf{x}]/J_3 \cap I_2 - 1\}.
\end{aligned}$$

Thus what we must prove is that the inequalities $\operatorname{depth} K[\mathbf{x}]/J_3 \geq 6$, $\operatorname{depth} K[\mathbf{x}]/I_2 \geq 6$ and $\operatorname{depth} K[\mathbf{x}]/J_3 \cap I_2 \geq 7$. Obviously, $\operatorname{depth} K[\mathbf{x}]/I_2 = 2k + 4 \geq 6$. Moreover, by Lemmas

3.3 (d) and 3.2 (b), we can easily see that $\operatorname{depth} K[\mathbf{x}]/J_3 \cap I_2 \geq (2k+5) - 2(k-1) = 7$. Thus we investigate $\operatorname{depth} K[\mathbf{x}]/J_3$.

**(First step)** By the short exact sequence

$$0 \to K[\mathbf{x}]/I_3 \cap I_4 \to K[\mathbf{x}]/I_3 \oplus K[\mathbf{x}]/I_4 \to K[\mathbf{x}]/(I_3 + I_4) \to 0,$$

one has

$$\begin{aligned}\operatorname{depth} K[\mathbf{x}]/J_1 &= \operatorname{depth} K[\mathbf{x}]/(I_3 + I_4)\\ &\geq \min\{\operatorname{depth} K[\mathbf{x}]/I_3, \operatorname{depth} K[\mathbf{x}]/I_4, \operatorname{depth} K[\mathbf{x}]/I_3 \cap I_4 - 1\}.\end{aligned}$$

By Lemma 3.2 (b), one has $\operatorname{depth} K[\mathbf{x}]/I_3 \geq k+6 \geq 6$ and $\operatorname{depth} K[\mathbf{x}]/I_4 \geq k+6 \geq 6$. Since $I_3 \cap I_4 = x_{k+1}x_{2k+2}x_{2k+3}(x_2, \ldots, x_k)(x_{k+3}, \ldots, x_{2k+1})$ by Lemma 3.3 (a) and $x_{k+1}, x_{2k+2}, x_{2k+3}$ do not appear in the elements of $G((x_2, \ldots, x_k)(x_{k+3}, \ldots, x_{2k+1}))$, one has $\operatorname{depth} K[\mathbf{x}]/I_3 \cap I_4 \geq (2k+5) - 2(k-1) = 7$ by Lemma 3.2 (b). Hence one has $\operatorname{depth} K[\mathbf{x}]/J_1 \geq 6$.

**(Second step)** Again, by the short exact sequence

$$0 \to K[\mathbf{x}]/J_1 \cap I_1 \to K[\mathbf{x}]/J_1 \oplus K[\mathbf{x}]/I_1 \to K[\mathbf{x}]/(J_1 + I_1) \to 0,$$

one has

$$\begin{aligned}\operatorname{depth} K[\mathbf{x}]/J_2 &= \operatorname{depth} K[\mathbf{x}]/(J_1 + I_1)\\ &\geq \min\{\operatorname{depth} K[\mathbf{x}]/J_1, \operatorname{depth} K[\mathbf{x}]/I_1, \operatorname{depth} K[\mathbf{x}]/J_1 \cap I_1 - 1\}.\end{aligned}$$

By Lemma 3.2 (a), $\operatorname{depth} K[\mathbf{x}]/I_1 \geq (2k+5) - 2(k-2) \geq 6$. Also by Lemma 3.3 (b), one has $J_1 \cap I_1 = x_{2k+3}(x_{k+1}, x_{2k+2})I_1$. Since only $2k-2$ variables appear in the elements of $G((x_{k+1}, x_{2k+2})I_1)$, and $x_{2k+3}$ does not appear in there, one has $\operatorname{depth} K[\mathbf{x}]/J_1 \cap I_1 \geq 7$ by Lemma 3.2 (b). In addition, one has $\operatorname{depth} K[\mathbf{x}]/J_1 \geq 6$ by the first step. Hence one has $\operatorname{depth} K[\mathbf{x}]/J_2 \geq 6$.

**(Third step)** Similarly, by the short exact sequences

$$0 \to K[\mathbf{x}]/J_2 \cap I_5 \to K[\mathbf{x}]/J_2 \oplus K[\mathbf{x}]/I_5 \to K[\mathbf{x}]/(J_2 + I_5) \to 0,$$

one has

$$\begin{aligned}\operatorname{depth} K[\mathbf{x}]/J_3 &= \operatorname{depth} K[\mathbf{x}]/(J_2 + I_5)\\ &\geq \min\{\operatorname{depth} K[\mathbf{x}]/J_2, \operatorname{depth} K[\mathbf{x}]/I_5, \operatorname{depth} K[\mathbf{x}]/J_2 \cap I_5 - 1\}.\end{aligned}$$

By Lemma 3.2 (b), one has $\operatorname{depth} K[\mathbf{x}]/I_5 \geq k+6 \geq 6$. For $\operatorname{depth} K[\mathbf{x}]/J_2 \cap I_5$, by Lemma 3.3 (c), one has $J_2 \cap I_5 = x_{k+2}x_{2k+2}x_{2k+4}(x_2, \ldots, x_k)(x_{2k+3}(x_2, \ldots, x_k) + I_1)$. Notice that only $2k-2$ variables appear and $x_{k+2}, x_{2k+2}, x_{2k+4}$ do not appear in the elements of $G((x_2, \ldots, x_k)(x_{2k+3}(x_2, \ldots, x_k) + I_1))$. Thus, again by Lemma 3.2 (b), one has $\operatorname{depth} K[\mathbf{x}]/J_2 \cap I_5 \geq 7$. Combining these results with the second step, one has $\operatorname{depth} K[\mathbf{x}]/J_3 \geq 6$.

Therefore, one has $\operatorname{depth} K[\mathbf{x}]/I \geq 6$, as required. $\square$

## 4. Cohen–Macaulayness of $K[\mathbf{x}]/\operatorname{in}_{<_{\mathrm{lex}}}(I_G)$

In this section, we will prove the following

**Lemma 4.1.** *Let $<_{\mathrm{lex}}$ denote the lexicographic order on $K[\mathbf{x}]$ induced by the ordering $x_1 > \cdots > x_{2k+5}$ of the variables. Then $K[\mathbf{x}]/\operatorname{in}_{<_{\mathrm{lex}}}(I_G)$ is Cohen–Macaulay.*

First of all, we need to know the generators of $\mathrm{in}_{<_{\mathrm{lex}}}(I_G)$. As an analogue of Lemma 1.1, we can prove the following

**Lemma 4.2.** *The set of binomials corresponding to primitive even closed walks (1), (2), (3) and (4) (appeared in section 1) is a Gröbner basis of $I_G$ with respect to $<_{\mathrm{lex}}$.*

**Corollary 4.3.** *The initial ideal of $I_G$ with respect to $<_{\mathrm{lex}}$ is generated by the following monomials:*

$$(\flat) \qquad \begin{aligned} & x_i x_{k+1+j}, \quad 2 \leq i < j \leq k, \\ & x_1 x_{k+2} x_{2k+4} x_{2k+5}, \\ & x_r x_{k+2} x_{2k+4}, \ x_1 x_{k+1+r} x_{2k+5}, \quad 2 \leq r \leq k. \end{aligned}$$

*In particular,* $\mathrm{in}_{<_{\mathrm{lex}}} I_G$ *is a squarefree monomial ideal.*

Note that we can exclude the initial term of the binomial corresponding to the even closed walk of type (4).

Let $I'$ be the initial ideal of $I_G$ with respect to $<_{\mathrm{lex}}$. Since $I'$ is squarefree, we can define a simplicial complex $\Delta'$ on $[2k+5]$ whose Stanley–Reisner ideal coincides with $I'$. In order to prove that $K[\mathbf{x}]/I'$ is Cohen–Macaulay, we will show that $\Delta'$ is shellable.

We recall the definition of the shellable simplicial complex. Let $\Delta$ be a simplicial complex. We call $\Delta$ is *pure* if every facets (maximal faces) of $\Delta$ have the same dimension. A pure simplicial complex $\Delta$ of dimension $d-1$ is called *shellable* if all its facets (those are all $(d-1)$-faces of $\Delta$) can be listed

$$F_1, F_2, \ldots, F_s$$

in such a way that

$$\Big(\bigcup_{j=1}^{i-1} \langle F_j \rangle\Big) \cap \langle F_i \rangle \quad \Big( = \bigcup_{j=1}^{i-1} \langle F_j \cap F_i \rangle \Big)$$

is pure of dimension $d-2$ for every $1 < i \leq s$. Here $\langle F_i \rangle := \{\sigma \in \Delta : \sigma \subset F_i\}$. It is known that if $\Delta$ is shellable, then $K[\Delta]$ is Cohen–Macaulay for any field $K$.

To show that $\Delta'$ is shellable, we investigate the facets of $\Delta'$. Let $F(\Delta')$ be the set of facets of $\Delta'$. Then the standard primary decomposition of $I' = I_{\Delta'}$ is

$$I_{\Delta'} = \bigcap_{F \in F(\Delta')} P_{\bar{F}},$$

where $\bar{F}$ is the complement of $F$ in $[2k+5]$ and $P_{\bar{F}} = (x_i : i \in \bar{F})$; see [2, Lemma 1.5.4]. Hence we can obtain $F(\Delta')$ from the standard primary decomposition of $I'$.

**Lemma 4.4.** *The standard primary decomposition of $I'$ is the intersection of the following prime ideals:*

$$(\sharp)\qquad \begin{aligned}
&(x_1)+(x_2,x_3,\dots,x_k),\ (x_{2k+5})+(x_2,x_3,\dots,x_k),\\
&(x_{k+2})+(x_{k+3},x_{k+4},\dots,x_{2k+1}),\ (x_{2k+4})+(x_{k+3},x_{k+4},\dots,x_{2k+1}),\\
&(x_1,x_{k+2})+I'_\ell,\ \ 2\le \ell\le k,\\
&(x_1,x_{2k+4})+I'_\ell,\ \ 2\le \ell\le k,\\
&(x_{k+2},x_{2k+5})+I'_\ell,\ \ 2\le \ell\le k,\\
&(x_{2k+4},x_{2k+5})+I'_\ell,\ \ 2\le \ell\le k,
\end{aligned}$$

*where $I'_\ell=(x_2,\dots,x_{\ell-1},x_{k+2+\ell},\dots,x_{2k+1})$ for $\ell=2,\dots,k$.*

*Proof.* Since there is no relation of inclusion among the prime ideals on ($\sharp$), it is enough to prove that the intersection of these prime ideals coincides with $I'$.

First, we consider the case where $k=1$. Then $G(I')=\{x_1x_3x_6x_7\}$ and ($\sharp$) consist of only the first 2 rows: $(x_1)$, $(x_7)$, $(x_3)$, and $(x_6)$. Thus the assertion trivially holds.

Next, we consider the case where $k=2$. Note that $I'_2=0$. Then the ideal $I'$ is

$$\begin{aligned}
I'&=(x_1x_4x_8x_9,x_1x_5x_9,x_2x_4x_8)\\
&=(x_1,x_2)\cap(x_1,x_4)\cap(x_1,x_8)\cap(x_4,x_5)\cap(x_4,x_9)\cap(x_8,x_5)\cap(x_8,x_9)\cap(x_9,x_2)\\
&=(x_1,x_2)\cap(x_9,x_2)\cap(x_4,x_5)\cap(x_8,x_5)\cap(x_1,x_4)\cap(x_1,x_8)\cap(x_4,x_9)\cap(x_8,x_9),
\end{aligned}$$

as desired.

Hence we may assume that $k\ge 3$. Then the intersection of the prime ideals on the first row of ($\sharp$) is

$$(x_1x_{2k+5},x_2,x_3,\dots,x_k)$$

and that on the second row of ($\sharp$) is

$$(x_{k+2}x_{2k+4},x_{k+3},x_{k+4},\dots,x_{2k+1}).$$

For $\ell=2,\dots,k$, the intersection of the prime ideals on the last 4 rows of ($\sharp$) is

$$\begin{aligned}
&((x_1,x_{k+2})+I'_\ell)\cap((x_1,x_{2k+4})+I'_\ell)\cap((x_{k+2},x_{2k+5})+I'_\ell)\cap((x_{2k+4},x_{2k+5})+I'_\ell)\\
=&((x_1,x_{k+2}x_{2k+4})+I'_\ell)\cap((x_{k+2}x_{2k+4},x_{2k+5})+I'_\ell)\\
=&(x_1x_{2k+5},x_{k+2}x_{2k+4})+I'_\ell.
\end{aligned}$$

Hence, the intersection of the prime ideals on the last 4 rows of ($\sharp$) for all $\ell$ is

$$(x_1x_{2k+5},x_{k+2}x_{2k+4})+\bigcap_{\ell=2}^{k} I'_\ell.$$

Therefore the intersection of all prime ideals of ($\sharp$) is

$$(1)\qquad \begin{aligned}
&x_1x_{2k+5}(x_{k+2}x_{2k+4},x_{k+3},x_{k+4},\dots,x_{2k+1})+x_{k+2}x_{2k+4}(x_1x_{2k+5},x_2,x_3,\dots,x_k)\\
&+(\bigcap_{\ell=2}^{k} I'_\ell)\cap(x_1x_{2k+5},x_2,x_3,\dots,x_k)\cap(x_{k+2}x_{2k+4},x_{k+3},x_{k+4},\dots,x_{2k+1}).
\end{aligned}$$

The ideal on the first row of (1) coincides with the one generated by monomials on the last 2 rows of ($\flat$). Since $I'_2 = (x_{k+4}, x_{k+5}, \dots, x_{2k+1})$ and $I'_k = (x_2, x_3, \dots, x_{k-1})$, the ideal on the second row of (1) coincides with $\bigcap_{\ell=2}^{k} I'_\ell$. Hence, we may prove that

$$\bigcap_{\ell=2}^{k} I'_\ell = (x_i x_{k+1+j} \ : \ 2 \le i < j \le k).$$

To show this equality, we prove

$$\bigcap_{\ell=2}^{k'} I'_\ell = (x_i x_{k+1+j} \ : \ 2 \le i < j \le k') + (x_{k+2+k'}, \dots, x_{2k+1}) \tag{2}$$

for $k' = 2, \dots, k$. When $k' = k$, we obtain the desired equality. We use induction on $k' \ge 2$. The case of $k' = 2$ is trivial. When (2) holds for $k'$, we have

$$\begin{aligned}
\bigcap_{\ell=2}^{k'+1} I'_\ell &= (\bigcap_{\ell=2}^{k'} I'_\ell) \cap I'_{k'+1} \\
&= ((x_i x_{k+1+j} \ : \ 2 \le i < j \le k') + (x_{k+2+k'}, \dots, x_{2k+1})) \cap (x_2, \dots, x_{k'}, x_{k+3+k'}, \dots, x_{2k+1}) \\
&= (x_i x_{k+1+j} \ : \ 2 \le i < j \le k') + x_{k+2+k'}(x_2, \dots, x_{k'}) + (x_{k+3+k'}, \dots, x_{2k+1}) \\
&= (x_i x_{k+1+j} \ : \ 2 \le i < j \le k'+1) + (x_{k+3+k'}, \dots, x_{2k+1}),
\end{aligned}$$

as desired. $\square$

Now we are in the position to prove Lemma 4.1.

*Proof of Lemma 4.1.* By Lemma 4.4, $F(\Delta')$ consists of the following subsets of $[2k+5]$:

$$\begin{aligned}
&F_1 = \overline{\{1\} \cup \{2, 3, \dots, k\}}, \ F_2 = \overline{\{2k+5\} \cup \{2, 3, \dots, k\}}, \\
&F_3 = \overline{\{k+2\} \cup \{k+3, k+4, \dots, 2k+1\}}, \\
&F_4 = \overline{\{2k+4\} \cup \{k+3, k+4, \dots, 2k+1\}}, \\
&G_{1,\ell} = \overline{A_1 \cup G'_\ell}, \ \ 2 \le \ell \le k, \\
&G_{2,\ell} = \overline{A_2 \cup G'_\ell}, \ \ 2 \le \ell \le k, \\
&G_{3,\ell} = \overline{A_3 \cup G'_\ell}, \ \ 2 \le \ell \le k, \\
&G_{4,\ell} = \overline{A_4 \cup G'_\ell}, \ \ 2 \le \ell \le k,
\end{aligned}$$

where $G'_\ell = \{2, \dots, \ell-1, k+2+\ell, \dots, 2k+1\}$ for $2 \le \ell \le k$, $A_1 = \{1, k+2\}$, $A_2 = \{1, 2k+4\}$, $A_3 = \{k+2, 2k+5\}$, $A_4 = \{2k+4, 2k+5\}$ and $\overline{F} = [2k+5] \setminus F$. Note that $G_{m,\ell} \cap A_j = \emptyset$ and $\#(G_{m,\ell}) = k-2$. In particular, $\Delta'$ is pure of dimension $k+4$.

Now we define the ordering on $F(\Delta')$ as follows:

$$G_{1,2}, \dots, G_{1,k}, G_{2,2}, \dots, G_{2,k}, G_{3,2}, \dots, G_{3,k}, G_{4,2}, \dots, G_{4,k}, F_1, F_2, F_3, F_4. \tag{3}$$

We will prove $\Delta'$ satisfies the condition of shellability with this ordering. For $F, G \in F(\Delta)$, we write $G \prec F$ if $G$ lies in previous to $F$ on (3).

First, we investigate $\Delta_{m,\ell} := (\bigcup_{G' \prec G_{m,\ell}} \langle G' \rangle) \cap \langle G_{m,\ell} \rangle = \bigcup_{G' \prec G_{m,\ell}} \langle G' \cap G_{m,\ell} \rangle$ for $m = 1, 2, 3, 4$. For $\ell' < \ell$, one has

$$\begin{aligned} G_{m,\ell'} \cap G_{m,\ell} &= \overline{A_m \cup G'_{\ell'}} \cap \overline{A_m \cup G'_\ell} \\ &= \overline{(A_m \cup G'_{\ell'}) \cup (A_m \cup G'_\ell)} \\ &= \overline{A_m \cup \{2, \ldots, \ell - 2, \ell - 1, k + 2 + \ell', k + 3 + \ell', \ldots, 2k + 1\}} \\ &\subset \overline{A_m \cup \{2, \ldots, \ell - 2, \ell - 1, k + 1 + \ell, k + 2 + \ell, \ldots, 2k + 1\}} \\ &= G_{m,\ell-1} \cap G_{m,\ell} \end{aligned}$$

and $G_{m,\ell-1} \cap G_{m,\ell}$ is a $(k+3)$-dimensional face. Then we can conclude that $\Delta_{1,\ell}$ is pure of dimension $k + 3$. Assume that $m = 2, 3, 4$. For $m' < m$, one has

$$\begin{aligned} G_{m',\ell'} \cap G_{m,\ell} &= \overline{A_{m'} \cup G'_{\ell'}} \cap \overline{A_m \cup G'_\ell} \\ &= \overline{(A_{m'} \cup G'_{\ell'}) \cup (A_m \cup G'_\ell)} \\ &\subset \overline{(A_{m'} \cup A_m) \cup G'_\ell}. \end{aligned}$$

When $m = 2$, then $m' = 1$ and

$$\overline{(A_1 \cup A_2) \cup G'_\ell} = \overline{\{1, k + 2, 2k + 4\} \cup G'_\ell} = G_{1,\ell} \cap G_{2,\ell},$$

which is $(k + 3)$-dimensional. Therefore we can conclude that $\Delta_{2,\ell}$ is a pure simplicial complex of dimension $k + 3$. Similarly, we can see that $\Delta_{m,\ell}$ is pure of dimension $k + 3$ for $m = 3, 4$ since e.g., $A_2 \cup A_3 \supset A_1 \cup A_3 = \{1, k + 2, 2k + 5\}$.

Next, we investigate $\Delta_s := \bigcup_{G \prec F_s} \langle G \cap F_s \rangle$ for $s = 1, 2, 3, 4$. It is easy to see that $G_{1,k} \cap F_1$ (resp. $G_{2,k} \cap F_1$) contains $G_{1,\ell} \cap F_1$ and $G_{3,\ell} \cap F_1$ (resp. $G_{2,\ell} \cap F_1$ and $G_{4,\ell} \cap F_1$). Thus facets of $\Delta_1$ are $G_{1,k} \cap F_1$ and $G_{2,k} \cap F_1$, those are $(k + 3)$-dimensional.

Similarly, we can see that the facets of $\Delta_2$ are $G_{3,k} \cap F_1$, $G_{4,k} \cap F_1$, and $F_1 \cap F_2$, those are also $(k + 3)$-dimensional.

For $\Delta_3$, we can verify that $G_{1,2} \cap F_3$ (resp. $G_{3,2} \cap F_3$) is a $(k + 3)$-dimensional face containing $G_{1,\ell} \cap F_3$, $G_{2,\ell} \cap F_3$ and $F_1 \cap F_3$ (resp. $G_{3,\ell} \cap F_3$, $G_{4,\ell} \cap F_3$ and $F_2 \cap F_3$). Therefore $\Delta_3$ is pure of dimension $k + 3$.

Similarly, we can see that $\Delta_4$ is also a pure simplicial complex of dimension $k+3$ whose facets are $G_{2,2} \cap F_4$, $G_{4,2} \cap F_4$, and $F_3 \cap F_4$. $\square$

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